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Noise and Superfluid Turbulence in He II: Theory

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Noise may be used as a probe of the dynamics of a nonlinear system. Weak noise provides information on the local dynamics near a transition. This information can be helpful for identifying the underlying bifurcation. The normal form of the bifurcation can then be used as a starting point for a phenomenological model of the system. Such an approach furnishes a unified explanation of the experimental data near the transition between superfluid turbulent states TI and TII in liquid helium counterflow. Strong external noise probes global features of the nonlinearities of a system. It may also dramatically affect the dynamical behavior of the system and give rise to noise-induced transitions. The problems of modeling this phenomenon in the liquid helium counterflow system are discussed and a first model, providing a unified description of the experimental observations for weak and strong noise, is developed.

KEY WORDS: Normal form; noise; thermal counterflow; TI–TII transition; noise-induced transition.

1. INTRODUCTION

One way to understand the rich temporal and spatial behavior of nonlinear systems is to study how simple behavior gives way to more complex behavior. From this viewpoint, instabilities of the systems and transitions between states with different dynamical behavior are the key to a description of nonlinear phenomena. The mathematical basis for such an approach is bifurcation theory. Bifurcation theory attempts to describe all

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of the ways in which dynamical systems can make transitions from one type of dynamical behavior to another. The simplest situation occurs if the system exhibits only transitions between different steady states. In this case, singularity theory[1] provides a very complete description of the different possibilities. The number of external control parameters that have to be adjusted to a particular numerical value for one of these different possible transitions to occur is the "codimension" of the bifurcation. We expect that low-codimension bifurcations will be most often encountered in typical systems, since fewer conditions have to be met. Bifurcation theory describes not only the types of transitions that can take the system from one type of behavior to another, but also furnishes the dynamics of the system in the neighborhood of the transition. It provides "normal forms" which are the simplest possible equations that contain the bifurcation in question.[2] The steady states of a general model can be related to those of the normal form via a smooth local transformation about the transition point. This is the very feature that makes bifurcation theory such a powerful tool to analyze and understand nonlinear systems. If we can properly identify the underlying bifurcation for an instability of a physical system, then the normal form provides a satisfactory model of the system, at least in the neighborhood of the instability. Since a complete theoretical description is generally only available for the simplest nonlinear systems, phenomenological models based on normal forms are often the only means to ask questions about the dynamics of the system, to organize experimental results, and to make new testable predictions.

Clearly, it becomes then an important task to correctly identify the underlying bifurcation for an experimentally observed instability. There are various methods available in nonlinear dynamics, based on time series analysis, phase-space reconstruction, Poincaré sections, etc., that help with this task. Here we will discuss yet another method based on an analysis of the fluctuations of the system in the vicinity of the instability. Irregular random fluctuations are unavoidable in macroscopic systems. They may be "internal fluctuations," due to the complex interaction of parts of a composite system, or they may be "external noise," due to irregular influences imposed on the system from the environment to which it is coupled. We will refer to both types of fluctuations as "noise." Noise will cause the state of the system to fluctuate. The way these fluctuations evolve as the system undergoes a transition can provide clues to the nature of the underlying bifurcation. Wiesenfeld[3] analyzed the effect of noise on the power spectra of systems close to instabilities of periodic orbits. He found that fluctuations add features to the spectra which are characteristic of the impending bifurcation. We have studied the behavior of fluctuations in the vicinity of transitions between steady states and applied our results[4] to the transition between superfluid turbulent states TI and TII in liquid helium counterflow experiments.[5–7]

The intensity of internal fluctuations generally scales with an inverse power of the system's size and these fluctuations are thus usually a weak source of disturbance in macroscopic systems. Obviously, external noise (unless deliberately applied) is usually also weak in laboratory experiments. Weak noise will give rise to Gaussian fluctuations in the state variable with small amplitude about the deterministic steady state. Thus, weak noise provides a natural probe of the local dynamics of a nonlinear system. Experimental studies of the effect of small-amplitude noise can therefore help with the identification of the underlying bifurcation of a transition in a nonlinear system. The results of such studies can be interpreted theoretically by using a linearized normal form. Our theoretical analyses of linearized normal forms perturbed by weak noise show that the behavior of the variance of the fluctuations about the steady state can furnish information about the transition and the nature of the underlying bifurcation.

If the noise strength is increased, further details of the dynamics of the system become accessible. Moderate noise causes the state of the system to leave the immediate vicinity of the transition point and to wander over a larger region of the state space. In this way, noise of moderate amplitude leads, so to speak, to an unfolding[1] of the normal form describing the bifurcation. Finally, if strong noise is imposed on the system, the noise drives the system even farther away from the vicinity of the transition point. In this way global features of the dynamics of the system are explored. Deliberately applied external noise can thus be used as a convenient probe of the global nonlinear dynamics of the system. Additionally, the interaction of the nonlinear dynamics of the system with strong external noise may give rise to noise-induced transitions,[8–12] which lead to drastic modifications of the dynamical behavior of nonlinear systems. Noise-induced transitions depend on the exact form of the nonlinearity; different nonlinear models of a given system that agree reasonably well with each other under deterministic external conditions often lead to markedly different predictions for the behavior of the system under fluctuating constraints.[13] Studies of noise-induced transitions, and in particular their associated "phase diagrams,"[12] can therefore help in finding a satisfactory model which goes beyond normal forms and incorporates global features of the dynamics.

2. WEAK NOISE

Tough and co-workers have studied the transition between two superfluid turbulent states of thermal counterflow in liquid helium. Their experiments have addressed the weak-noise[5,6] and the strong-noise case.[7] The essential piece of their apparatus is a reservoir of liquid helium connected to a small chamber with a heater by means of a thin flow tube. (For more details see refs. 5–7.) The state of the system can be characterized by the chemical potential difference $\Delta\mu$ across the tube. This state variable is proportional to the vortex line density L in the superfluid flowing through the tube. The behavior of the state variable $\Delta\mu$ (or L) is measured as a function of $\dot{Q}$, the rate at which heat is introduced into the system by the heater. According to the two-fluid model, the flow through the tube may be treated as a counterflow of a normal fluid component, flowing from the heater to the reservoir, and a superfluid component, flowing in the opposite direction. For low values of $\dot{Q}$ the flow of both components is laminar and the two components do not interact; there are no vortices in the superfluid, $L = 0$, and the chemical potential difference $\Delta\mu$ across the tube vanishes. At a critical value of $\dot{Q}$ the laminar state ceases to be globally stable. A turbulent state, which corresponds to the appearance of quantized vortices in the superfluid component ($L \neq 0$), can now be sustained by the system. This turbulent state is denoted TI. The laminar state remains locally stable (metastable) and a finite-amplitude perturbation is required for the system to "nucleate" vorticity and to make the transition to the turbulent state TI. In the state TI the two components experience a mutual friction and the chemical potential difference across the tube is no longer zero. As $\dot{Q}$ is further increased, the state TI undergoes a complex transition to another superfluid turbulent state, denoted TII.

A microscopic theory exists for the turbulent state TII.[14] This state can be understood in terms of a homogeneous tangle of vortex lines. Schwarz has derived an equation that describes the motion of a vortex in terms of its local radius of curvature and normal and superfluid velocity fields. When combined with rules concerning the behavior of vortex lines in close proximity to each other, numerical simulations yield results in excellent agreement with experiments for the TII state. No such successful treatment is available for the state TI and the transition between the two turbulent states.

The turbulent superfluid states TI and TII are macroscopic steady states. We have made progress in the understanding of the TI–TII transition by combining experimental observations on the behavior of the fluctuations in the state of the system with ideas from bifurcation theory.[4] The TI–TII transition point is defined by the fact that the relaxation time passes through a maximum, i.e., it is the point where the state of the system is the least stable. Though, by definition, the relaxation time is enhanced at the transition point, experiments show no indication that it diverges. In other words, the system passes through a region of weakened stability, but no actual instability is encountered. However, further experimental studies, in which the resolution is varied, are necessary to establish unambiguously that the relaxation time does not diverge at the transition point; finite resolution in the experiments would mask divergent behavior of the system.[4]

This is the first important experimental clue for identifying the underlying bifurcation and writing a normal form to describe the dynamics in the vicinity of the TI–TII transition. A bifurcation between steady states occurs if the Jacobian of the system, evaluated at the steady state, has a singularity, i.e., zero eigenvalue. This implies that the steady state loses stability and that the relaxation time goes to infinity. However, as mentioned above, depending on the codimension of the bifurcation, a certain number of auxiliary parameters have to be adjusted to particular numerical values for the bifurcation to occur as the bifurcation parameter is varied. If these secondary parameter deviate slightly from these particular values, no singularity of the Jacobian and no bifurcation occur. However, by continuity, an eigenvalue will pass close to zero. Thus, the system will pass through a region of weakened stability. Such a continuous transition between two steady states, in which no singularity is encountered, is known as an "imperfect" bifurcation. The experimental data on the behavior of the relaxation time at the TI–TII transition suggest that an imperfect bifurcation occurs in the counterflow system.

The second clue is contained in the data on the power of the fluctuations at 0.1 Hz. This power can be taken as a measure for the variance or strength of the fluctuations about the steady state of the system. We expect the fluctuations to be enhanced at the point of weakened stability, i.e., at the transition point. However, the system behaves in a more complicated and intriguing way. For the experiments at 1.6 K the strength of the fluctuations shows a narrow maximum at the TI–TII transition point, but, most surprisingly, there is another broad maximum to the left of the transition point, i.e., for lower values of $\dot{Q}$, which actually dominates the peak corresponding to the transition point. The point where the second maximum occurs shows no special features at all in the steady-state diagram or in the plot of relaxation times. Any phenomenological model, in order to be satisfactory, must be able to account for this unexpected feature of the fluctuations. The situation is even more intriguing for the experiments at 1.75 K; there is no clearly resolved local maximum at the transition point at all. This imposes additional constraints on the phenomenological model. It should be able to describe the change in the behavior of the fluctuations as the temperature of the reservoir is changed.

[4] We are indebted to R. Ecke for this remark.

As mentioned above, we have used bifurcation theory to organize the experimental data and to understand the TI–TII transition. The simplest transitions between macroscopic steady states are one-state-variable bifurcations. Certain transitions between steady states do have more than one state variable in their normal form; however, these all have codimension 3 or greater.[1] In this sense they are less likely to be encountered in a physical system. Therefore, it is quite natural to attempt to model the transition between the state TI and the state TII using only a single state variable. Furthermore, many properties of the turbulent superfluid states, in particular of TII, are successfully described by a phenomenological one-state-variable theory based on the Vinen equation.[15,16] The Vinen equation, however, does not describe the TI–TII transition.

We have suggested that the transition between states TI and TII takes place via an imperfect pitchfork bifurcation perturbed by both additive and linear multiplicative noise.[4] The pitchfork bifurcation is the only codimension-2 bifurcation that mediates a continuous transition between steady states. Additionally, this bifurcation contains in its unfolding the transcritical and hysteresis bifurcations, which are the only codimension-1 bifurcations giving rise to continuous transitions between steady states.[1] A normal form for the pitchfork bifurcation is $g(x, \lambda) = -\lambda x - x^3$. [Signs are chosen so that the well-known "pitchfork" diagram of steady states given by the roots of $g(x, \lambda) = 0$ opens out to the left. This facilitates comparison with the experimental steady-state curve for the counterflow; see ref. 4.] In the normal form, x characterizes the state of the system, generally a deviation from some reference state, and λ is a distinguished parameter, the "bifurcation parameter." The pitchfork bifurcation has codimension 2, so that two auxiliary parameters must be adjusted in order to achieve the singularity at $x = 0$, $\lambda = 0$. Small perturbations will generally destroy this singularity. However, close to the singularity all analytic perturbations are equivalent to those generated by two terms with coefficients a_0 and a_2 in the "universal unfolding" of the normal form $g(x, \lambda)$ of the pitchfork:

$$p(x, \lambda) = a_0 - \lambda x + a_2 x^2 - x^3 \tag{1}$$

Steady states $x(\lambda)$ of the unfolded bifurcation are given by $p(x(\lambda), \lambda) = 0$ and the dynamics are modeled close to the singularity by $\dot{x} = p(x, \lambda)$. The codimension-2 pitchfork bifurcation is at the origin of the (a_0, a_2) parameter plane, where the two secondary parameters are equal to a particular numerical value, $a_0 = 0$ and $a_2 = 0$. The line of codimension-1 transcritical bifurcations is given by $a_0 = 0$. The curve $a_0 = a_2^3/27$ is the loci of codimension-1 hysteresis bifurcations.

If the secondary parameters are not adjusted to these particular values, no singularity and no bifurcation occur. However, when the deviation of the secondary parameters from the particular values necessary for a singularity is small, then the system will pass, as the bifurcation parameter λ is varied, through a region of weakened linear stability and an imperfect bifurcation occurs, as already discussed above. In the unfolding of the pitchfork bifurcation, these imperfect bifurcation have the character of imperfect transcritical bifurcations or of imperfect hysteresis bifurcations. Near an imperfect bifurcation, i.e., in the region of weakened stability, the magnitude of fluctuations about the steady state and their relaxation times will be enhanced. However, since no true singularity is encountered, the relaxation times will always remain finite, i.e., fluctuations will decay exponentially. We call the point at which the relaxation time reaches its greatest value the "paracritical" point.

As mentioned above, much may be learned about the dynamics of a system undergoing a bifurcation by analyzing fluctuations. Wiesenfeld has thoroughly studied fluctuations in systems bifurcating from limit cycles.[3] An analysis of fluctuations associated with imperfect bifurcations between steady states has allowed us to show that an imperfect pitchfork bifurcation is indeed a satisfactory description of the TI–TII transition.[4] Our model has a deterministic part, based on the normal form (1), and includes two sources of noise:

$$\begin{aligned} dx &= [a_0 - (\lambda - z)x + a_2 x^2 - x^3]\, dt + \sigma_1\, dW_1 \\ &= p(x, \lambda)\, dt + \sigma_1\, dW_1 + xz\, dt \end{aligned} \tag{2}$$

$$dz = -\gamma z\, dt + \sigma_2\, dW_2 \tag{3}$$

This model contains the two sources of noise to which any nonequilibrium system will inevitably be subject, namely internal noise or thermal noise, which represents the influence of the large number of (microscopic) degrees of freedom on the behavior of the system. Internal noise generally evolves on a time scale very fast compared to the time scale of the system and is thus usually modeled by an additive Gaussian white noise. This is the term $\sigma_1\, dW_1$ in (2). Nonequilibrium systems are open systems and as such are coupled to an environment. The fluctuations in the environment are a second source of noise for the system. The effect of these fluctuations on a nonlinear system is often state dependent, for instance, if they give rise to noise in the bifurcation parameter. In our model this leads to the linear term $xz\, dt$, modeling a source of random disturbances perturbing the bifurcation parameter λ. Since the ratio of the time scales of the system and the external noise can change as the paracritical point is approached, we have chosen to represent this linear noise by a Gaussian process with a non-vanishing correlation time, namely an Ornstein–Uhlenbeck process

(OU noise) with correlation time γ^{-1}. This allows us to describe the competition between the two time scales close to the paracritical point.

In the case of weak noise, we can linearize (2) about the deterministic steady state $x(\lambda)$ and calculate the variance v of the fluctuations about it. The variance v is given by the sum of two terms, $v = v_1 + v_2$, one due to the additive internal fluctuations and the other due to the multiplicative noise.

An analysis of our model (2) in the near-transcritical limit shows that the paracritical point occurs close to the point of maximum curvature of the curve of steady states $x(\lambda)$. The contribution v_1 of the additive noise to the total variance is proportional to the deterministic relaxation time, and thus reaches its maximum value at the paracritical point. However, the contribution v_2 of the multiplicative noise rises to the left of the paracritical point. It turns out that, depending on the values of the auxiliary parameters a_0 and a_2, two types of behavior are possible for the total variance v: (1) The total variance has a narrow maximum at the paracritical point, corresponding to v_1, and a broad maximum to the left of the paracritical point, corresponding to v_2. (2) The total variance has only one broad maximum, which occurs to the left of the paracritical point; there is no maximum at the paracritical point itself.

If the parameter values a_0 and a_2 are chosen near the curve of hysteresis bifurcations, the paracritical point is close to the inflection point in the curve $x(\lambda)$. For this case of an imperfect hysteresis bifurcation, analysis shows that the maxima of v_1 and v_2 are only slightly separated and both occur close to the paracritical point. The total variance always has only a single peak.

A detailed comparison of our model results with the experimental data shows that the TI–TII transition can be understood in terms of an imperfect pitchfork bifurcation.[4] In particular, the system at 1.6 K undergoes an imperfect transcritical bifurcation, as indicated by the location of the paracritical point on the curve of steady states and the two maxima in the power of fluctuations. Use of the same criteria leads to the conclusion that at 1.75 K the system goes through an imperfect bifurcation that corresponds to a crossover regime between a transcritical and a hysteresis bifurcation. Both bifurcations are in the unfolding of the pitchfork bifurcation. The universal unfolding of this bifurcation, appended by random forcing terms to model the noise sources of the system, provides thus the first unified explanation for the parametric behavior of the steady states, the characteristic time of relaxation to those steady states, and the power in fluctuations about the steady states near the TI–TII transition in turbulent thermal counterflow in He II.

The model also predicts a branch of unstable states and another branch of stable steady states which have not yet been observed. However, the normal form (1) represents a power-series expansion of the pitchfork bifurcation about the singular point, which is close to the paracritical point. Therefore, our model will provide a satisfactory description of the system only in some neighborhood of the TI–TII transition point. Fitting our model to the experimental data, we find that for the imperfect hysteresis bifurcation, the second branch of stable steady states never passes close to the paracritical point and may well lie outside the region where (2) is an adequate model of the TI–TII transition. However, in the case of the imperfect transcritical bifurcation the branch of unstable steady states closely approaches the paracritical point. Thus, we predict that, close to the transition point, the response of the system to perturbations whose strength exceeds some critical value will change qualitatively. Indeed, if a perturbation carried the system across the line of unstable steady states, the system must relax to a different, metastable state, *if* the dynamics continues to be described by a single state variable. If the system relaxes back to the original stable state, it must do so through a mechanism described by more than one state variable. The unstable steady states might then give rise to some anomaly in the relaxation time. In any case, if a study is made of the response to perturbations for the 1.6 K system near the TI–TII transition point, we predict that there will be a perturbation of critical strength, beyond which the dynamics changes.

Our analysis produces good qualitative agreement with experiments, but it has two limitations which should not be overlooked. First, it is limited to weak noise, since it based on the linearization of the normal form. Thus, it is not directly applicable to the moderate- or strong-noise case. For moderate noise the full nonlinear normal form needs to be considered. However, this is only a technical problem and the theoretical treatment of the effects of moderate noise requires only a straightforward extension of our methods. The situation is quite different for the strong-noise case. Strong noise will drive the system far away from the paracritical point into regions of state space where even the nonlinear normal form is no longer an adequate description of the dynamics. Griswold and Tough have conducted experiments with strong external noise and we will come back to this problem. Second, we do not take into account the spatial extent of the narrow tube through which the helium flows. Temporal fluctuations at one end of the tube may be carried by the flow and therefore become spatial fluctuations. The measured state variable, the chemical potential difference between the ends of the tube, must represent some average of the local fluctuations. The good qualitative agreement between our theoretical description and the experimental observations suggests that these spatial fluctuations do not strongly affect the dynamics of the measured state variable, at least in the weak-noise case.

3. STRONG EXTERNAL NOISE

Recently Griswold and Tough have reported that strong external noise can have a dramatic effect on superfluid turbulence in thermal counterflow.[7] In the weak-noise case, there exists only one superfluid turbulent state for a given value of $\dot{Q}$. Strong external noise modifies the TI–TII transition and induces bistability between a turbulent state with a low vortex line density and the state TII for a certain range of $\dot{Q}$. We conjecture that the first state corresponds to the unstable state (see below); strong noise apparently suppresses the TI state and stabilizes, in a stochastic sense, the unstable state. This represents the first experimental observation of a *noise-induced transition* to bistability in a physical system. The existence of this phenomenon was predicted a decade ago,[10] based on studies of a simple model system, the genetic model defined by

$$\dot{x} = 0.5 - x + \lambda x(1 - x) \tag{4}$$

Here x varies between 0 and 1 and the external parameter λ varies between $-\infty$ and ∞. For a detailed discussion of this model and its biological significance see ref. 12. The steady states of (4) are given by

$$x_{ss} = (2\lambda)^{-1}\,[\lambda - 1 + (1 + \lambda^2)^{1/2}] \tag{5}$$

In other words, for each value of the external parameter λ there exists a unique steady state, which is globally stable, as is easily verified. Thus, the genetic model (4) does not display any transition for deterministic external constraints. If noise is applied to the external parameter, $\lambda \to \lambda + \sigma\zeta$, then the state of the system is described by a probability density $p(x, t)$ and steady states correspond to a stationary probability density $p_{ss}(x)$ (ζ is a zero-mean, unit-variance stationary random process). We have argued[12] that a transition for a system subject to noise corresponds to a qualitative change in $p_{ss}(x)$. We have also argued[12] that the extrema x_m of $p_{ss}(x)$ are the appropriate indicators of a qualitative change in the stationary probability density and hence of a transition in the behavior of the system. Further, the maxima are the most probable states which are preferentially observed in experiments; they correspond to the states around which the system fluctuates. If the external parameter λ of the genetic system is perturbed by Gaussian white noise, then the extrema of $p_{ss}(x)$ are given by[10,12]

$$0.5 - x_m + \lambda x_m(1 - x_m) - (\sigma^2/2)\, x_m(1 - x_m)(1 - 2x_m) = 0 \tag{6}$$

Consider the case that the external parameter has a zero mean, $\lambda = 0$. Then the deterministic steady state is given by $x_{ss} = 0.5$. From (6) we find for the system perturbed by external noise

$$x_{m,0} = 0.5$$

and

$$x_{m,\pm} = \{1 \pm [1 - (4/\sigma^2)]^{1/2}\}/2 \qquad \text{for} \quad \sigma^2 \geqslant \sigma_c^2 = 4$$

Thus, the system with external white noise has a noise-induced critical point at $\lambda_c = 0$, $\sigma_c^2 = 4$, and $x_c = 0.5$.[10,12] For $\sigma^2 < 4$, the stationary probability density has a single peak centered on $x_{m,0} = 0.5$. At $\sigma_c^2 = 4$, this maximum becomes a double maximum and for $\sigma^2 > 4$ it splits in two; the probability density becomes double-humped. The external noise has induced bistable behavior! Recall that for deterministic external constraints no bistability is possible. This noise-induced transition is robust and occurs also for colored noise, i.e., noise with a nonzero correlation time.[12] The genetic model with external noise is a simple illustration of the surprising fact that external multiplicative noise, in addition to disorganizing the system, can also impose a structure that is not present without the noise.[12]

Though the experiments of the effect of external noise on superfluid turbulence confirm qualitatively the prediction of the genetic model as to the existence of noise-induced critical points, a quantitative theoretical treatment of the noise-induced bistability of the TI and TII states is no simple matter. We face several complications:

1. We are now concerned with external noise which involves a large range of heat currents $\dot{Q}$ and spreads the probability distribution for the system over a wide interval in the domain of the state variable. The dynamics is no longer dependent on only the local character of the TI–TII transition; global aspects of the system come into play. So our task is now to find a simple model that satisfactorily describes the essential feature of the weak-noise case as well as those of the strong-external-noise case.

Our approach is based on the following assumptions.[17] Even in the strong-noise case the superfluid turbulence is still described by one state variable. As Griswold and Tough[7] remark, there is no experimental evidence to the contrary (however, see below). Since we are looking for a unified model that contains the weak- and strong-noise cases, our dynamical equation is based on the unfolded normal form (2) of the pitchfork bifurcation. However, it is amended in several way in order take account of several features of the global dynamics revealed by the strong-noise studies:

$$\dot{x} = f(x, \lambda) = k(L^{1/2})\, m(\lambda)\, p(x, \lambda) \tag{7}$$

The factor $k(L^{1/2})$ is the mechanism by which we introduce the laminar steady state, $L=0$, and the unstable steady state that must exist between the turbulent and laminar steady states. As mentioned above, experiments show that the counterflow system can be maintained in a metastable laminar state well past the critical heat current of the TI–TII transition. This is a global feature of the dynamics and as such is obviously not contained in the normal form (2), which is valid only locally near the TI–TII transition point. The laminar steady state must correspond to a zero of $f(x, \lambda)$ when $L=0$. This zero is supplied by $k(L^{1/2})$. We have chosen for $k(L^{1/2})$ the following form:

$$k(L^{1/2}) = 1 - a/[1 + (L^{1/2} - L_0^{1/2})^2/w^2]$$

For further details see ref. 17.

The function of the factor $m(\lambda)$ is to bias the system toward greater relaxation times as λ increases. It has the general form

$$m(\lambda) = 1 + c_1\lambda + c_2\lambda^2$$

The motivation for this factor comes from the measured relaxation times for the counterflow system as shown in Fig. 64 of Griswold's dissertation.[18] The experimental data show a sharp peak in the relaxation time at the paracritical point, as does the pitchfork (2). For larger values of λ, however, the relaxation time soon begins to increase linearly with λ. We found no noise-induced bistability in our model if the bias factor $m(\lambda)$ was omitted.

2. In the strong-noise experiments, the current through the heater is a random current with an average dc value. In accordance with Ohm's law, the random current must be squared to give the random heat current acting on the system. One effect of this squaring is to modify the average amount of heat generated as compared with that estimated from the average current through the resistor. This effect has already been taken into account by Griswold and Tough in reporting their data. Their average heat currents include the effect of squaring the random component. The other effect of the squaring is to change the probability distribution of the noise. If the random component of the electrical current originally had a Gaussian distribution, the random component in the heat output will have a somewhat narrower distribution. In our preliminary studies we have ignored this effect. Our analysis aims only at qualitative agreement with the strong-noise experiments in a first stage, and furthermore, there is a second and more serious source of nonlinearity in the problem. This nonlinearity of the noise is caused by the fact that the normal variables of model (2) are

related to the physical variables L and $\dot{Q}$ by the following coordinate transformation[17]:

$$x = L^{1/2} - s\dot{Q} + b, \qquad \lambda = \dot{Q} - q_0$$

Here s is the slope of the line of steady states TII, $-b$ is the intercept of the continuation of the line of TII states with the $L^{1/2}$ axis, and q_0 is the heat current at the TI–TII transition point. This coordinate transformation has the consequence that powers of the external noise up to fifth order appear in the model equation. This nonlinearity makes the analysis of the problem very interesting from the point of view of stochastic methods. It requires an extension of the wideband perturbation method[12] in order to deal with the problem in the short-time correlation limit. Using a wideband perturbation analysis, we obtain a Markovian diffusion process that describes the system in the short-correlation-time limit. This limit is relevant for the superfluid turbulence experiments, since the correlation time of the external noise is about an order magnitude smaller than the relaxation times of the system.

3. The data for the external noise experiments show that the stationary probability density always decreases to zero at the laminar state, even though in many cases it reaches a sharp maximum at a low vortex line density very close to the laminar state. This was true for external noises whose standard deviations were 15% and 49% of the paracritical heat current. Since the laminar state is locally stable and exists for a large range of values of the heat current, in fact, as already mentioned for values well beyond the paracritical value, and since the state variable L is independent of $\dot{Q}$, the external noise will not kick the system out of the laminar state. It is only the very weak internal noise that could drive the system out of the stable laminar state. Thus, the Markovian diffusion process should spend a very large fraction of its time in the laminar state. Probability should accumulate there and produce a sharp maximum in the stationary probability density. Clearly, the experimental data imply that some feature in the dynamics destabilized the laminar state. A good candidate is random spatial inhomogeneities. We expect these inhomogeneities to play an important role if the average vortex line density is small, whereas they should have a negligible effect near the turbulent states with finite average vortex line densities. This question needs, however, further theoretical and experimental study. In any case, the experiments indicate that description in terms of a single state variable is unsatisfactory for strong noise in the neighborhood of the laminar state, whereas it seems to be valid everywhere else as remarked above. A way to reconcile these observations is to prescribe special boundary conditions for the Markovian diffusion process near $L = 0$. We have chosen *reinjection boundary* conditions: When a

realization reaches the state $L = 0$, it is immediately sent back to a finite value of L. In this way, the theoretical description keeps its one-state-variable character. The fact that other state variables become important near the laminar state is modeled by the reinjection process. This process captures the essential role of the additional state variables, namely the destabilization of the laminar state which prevents probability from accumulating there.

A model based on the above assumptions and considerations provides a first attempt to describe the superfluid turbulence in a unified way for both weak and strong noise. In the weak-noise limit it reduces, by construction, to the dynamical equation (2). In the strong-noise regime, it displays noise-induced bistability. However, the strong-noise results are preliminary so far, and further studies are required to confirm that (7), combined with reinjection boundary conditions, is a satisfactory model for superfluid turbulence with strong external noise.

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